J. Appl. Maths Mechs, Vol. 55, No. 4, pp. 517-522, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

EVOLUTION AND RESONANCES IN THE ROTATIONAL MOTION OF A VISCOELASTIC PLANET IN AN ELLIPTICAL ORBIT*

E.V. SINITSYN

Bolshevo, Moscow Region

(Received 26 November 1990)

The phenomenon of tidal evolution in the motion of planets in the solar system is studied using the many-parameter model of the linear theory of viscoelasticity. The planet is modelled as an almost spherical viscoelastic body whose centre of mass moves in an elliptical orbit. The method of averaging is used to obtain the equations describing the evolution of plane rotational motion in the non-resonant and resonant cases. The limiting angular velocity and conditions for the existence of a stable resonant rotation are obtained as functions of the eccentricity and of the parameters characterizing the properties of the material of the planet. The ranges of variation of these parameters for which the results agree with those obtained earlier on the basis of the hypotheses concerning the structure of the momentum of tidal forces /1, 2/, or by using the Kelvin-Voigt model for the planet material and introducing additional assumptions about its properties /3, 4/ are indicated.

1. Let us consider a homogeneous isotropic viscoelastic body (a planet) of density ρ made of material with the following properties: Poisson's ratio ν is independent of time, and the constitutive relations of the linear theory of viscoelasticity under small deformations /5-7/ are given in the form

$$\sigma_{kk} = \frac{2(1+\nu)}{1-2\nu}\left(E\varepsilon_{kk} + \eta\dot{\varepsilon}_{kk} + \int_0^t \mu(t-\xi)\frac{\partial e_{kk}}{\partial\xi}d\xi\right) \tag{1.1}$$

$$s_{ij} = 2\left(Ee_{ij} + \eta\dot{e}_{ij} + \int_0^t \mu(t-\xi)\frac{\partial e_{ij}}{\partial\xi}d\xi\right)$$

Here σ_{kk}, ε_{kk} and s_{ij}, e_{ij} $(i, j = 1, 2, 3)$ are the spherical parts and components of the stress and deformation tensor deviators, respectively, E and η are constants characterizing the viscoelastic properties of the material and $\mu(t-\xi)$ is the relaxation function which can be written in the form /5/

$$\mu(t-\xi) = \sum_{l=1}^{M} G_l \exp\left(-\frac{t-\xi}{\tau_l}\right) \tag{1.2}$$

where G_l, τ_l are material constants determined experimentally.

Let $O\xi_1\xi_2\xi_3$ be the inertial coordinate system with origin at the centre of attraction, and let the centre of mass C of the planet describe an elliptical orbit about the point O with eccentricity e, lying in the plane $O\xi_1\xi_2$ (the $O\xi_1$ axis is directed towards the perimeter). Let us attach to the plant a "mean" system of coordinates $Cx_1x_2x_3$ in accordance with the conditions (see e.g. /8/)

$$\int \mathbf{u}(\mathbf{r}, t)\,\rho dx = 0, \quad \int \mathbf{r}\times\mathbf{u}(\mathbf{r}, t)\,\rho dx = 0, \quad dx = dx_1dx_2dx_3$$

where $\mathbf{r} = (x_1, x_2, x_3)^T$ is the radius vector of the point of the undeformed body, and $\mathbf{u}(\mathbf{r}, t)$ is its displacement. Here and henceforth the integration will be carried out over the region occupied by the body in its undeformed state.

*Prikl.Matem.Mekhan.,55,4,647-652,1991

We introduce the following notation: $\mathbf{R} = OC$, $\boldsymbol{\omega}$ is the angular velocity of the trihedron $Cx_1x_2x_3$, $\boldsymbol{\gamma} = \mathbf{R}\,|R|^{-1}$, and a_i is the projection of the vector a on the Cx_i $(i = 1, 2, 3)$ axis.

We will assume that the motion of the centre of mass of the planet is independent of the motion about the centre of mass itself. Then

$$\frac{\mu}{R^3} = \frac{\omega_0^2}{(1-e^2)^3}(1 + e\cos\vartheta)^3$$

where μ is the gravitational constant, ω_0 is the orbital angular velocity of the centre of mass in averaged motion, and $\vartheta(t)$ is the true anomaly:

$$\vartheta^{\cdot} = \frac{\omega_0}{(1-e^2)^{3/2}}(1 + e\cos\vartheta)^2 \tag{1.3}$$

The potential energy of gravitational forces can be written, apart from terms $(L/R)^2$ (L is the characteristic dimension of the planet) in the form

$$\Pi = -\frac{\mu}{R}\int\left(1 - \frac{(\mathbf{r}+\mathbf{u}, \boldsymbol{\gamma})}{R} - \frac{(\mathbf{r}+\mathbf{u})^2}{2R^2} + \frac{3(\mathbf{r}+\mathbf{u}, \boldsymbol{\gamma})^2}{2R^2}\right)\rho\, dx$$

Let us construct a system of equations describing the motion of the axes $Cx_1x_2x_3$ and the deformation of the planet.

The variation in the principal angular momentum relative to the centre of mass is described by

$$\mathbf{K}^{\cdot} + \boldsymbol{\omega}\times\mathbf{K} = \mathbf{M} \tag{1.4}$$
$$\mathbf{K} = \int(\mathbf{r}+\mathbf{u})\times(\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u}) + \mathbf{u}^{\cdot})\rho\, dx$$
$$\mathbf{M} = \boldsymbol{\gamma}\times\operatorname{grad}_{\boldsymbol{\gamma}}\Pi$$

and we write the equations of motion of the continuum in the form

$$u_i^{\cdot\cdot} = \rho^{-1}\sigma_{ij,j} + F_i \tag{1.5}$$
$$\mathbf{F} = -\boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})) - \boldsymbol{\omega}^{\cdot}\times(\mathbf{r}+\mathbf{u}) - 2\boldsymbol{\omega}\times\mathbf{u}^{\cdot} - \rho^{-1}\operatorname{grad}_{\mathbf{u}}\Pi$$

where σ_{ij} is the stress tensor and F is the mass force field. The boundary of the body is stress-free.

The system (1.3)-(1.5) is closed by the kinematic Poisson's equations.

2. Let us assume that the period of free elastic oscillations of the planet and the time of their decay are much smaller than the period of rotation of the centre of mass along the orbit. We introduce a small parameter $\varepsilon\ (0 < \varepsilon \ll 1)$. If Ω_1 is the lowest frequency of free elastic oscillations $(\eta = 0,\ \mu(t-\xi) \equiv 0)$, then $\varepsilon = (\omega_0/\Omega_1)^2$.

Let us put in (1.1) and (1.2)

$$E = \frac{1}{\varepsilon}E^{\circ}, \quad \eta = \frac{1}{\varepsilon}\eta^{\circ}, \quad \mu(t-\xi) = \frac{1}{\varepsilon}\mu^{\circ}(t-\xi), \quad G_l = \frac{1}{\varepsilon}G_l^{\circ} \tag{2.1}$$

The quantities $|\boldsymbol{\omega}|$ and ω_0 are assumed to be of the same order. The assumption that Poisson's ratio is constant enables us to seek the displacements in the form /5/

$$\mathbf{u} = \sum_{n=1}^{\infty} q_n(t)\,\mathbf{U}_n(\mathbf{r}) \tag{2.2}$$

where $\mathbf{U}_n(\mathbf{r})$ are the characteristic forms of elastic oscillations.

Let us substitute the series (2.2) into (1.4) and (1.5) and make the change of variables $q_n = \varepsilon q_n^*$ $(\mathbf{u}(\mathbf{r}, t) = \varepsilon\mathbf{u}^*(\mathbf{r}, t))$. This yields a system of singularly perturbed integrodifferential Volterra-type equations. It can be shown that an asymptotic form can be constructed using the method of boundary functions /9/. Neglecting the exponentially decaying past (with the exponential index of the order of ε^{-1}) of the asymptotic expansion of the boundary layer-type, we find that the equations for determining q_n^* (taking into account the terms in ε^0), describe a quasistatic process of deformation. We shall further assume that the surface of the planet is nearly spherical and has the following form in spherical coordinates:

$$r = r_0 + \varepsilon f(\varphi, \psi) \tag{2.3}$$

where f is a function whose explicit form will not be needed in what follows.

Using the algorithm for constructing the asymptotic expression /10/ for a surface of the form (2.3), we find that the displacement field represents, within the limits of accuracy in ε, a displacement field of a viscoelastic sphere of radius r_0.

The planet may move such that the Cx_3 axis of the "mean" coordinate system is orthogonal, at all times, to the plane of the orbit /8/. Eq.(1.4), taking the terms linear in ε into account, then takes the form

$$\begin{aligned}\omega_3^{\cdot} = 3\frac{\mu(A-B)}{R^3C}\gamma_1\gamma_2 - \frac{\varepsilon}{C}\Big(\mathbf{e}_3, \frac{3\mu}{R^3}\gamma\int((\mathbf{u}^*, \gamma)\mathbf{r} + (\mathbf{r}, \gamma)\mathbf{u}^*)\rho\, dx + \\ \frac{d}{dt}\int(\mathbf{u}^*\times(\omega\times\mathbf{r}) + \mathbf{r}\times(\omega\times\mathbf{u}^*))\rho\, dx + \\ \omega\times\int((\mathbf{u}^*\times(\omega\times\mathbf{r}) + \mathbf{r}\times(\omega\times\mathbf{u}^*))\rho\, dx)\end{aligned} \tag{2.4}$$

where A, B, C are the principal central moments of inertia of the undeformed body, $\mathbf{e}_3$ is the unit vector of the axis Cx_3, and the quantity $(A - B)/C = O(\varepsilon)$.

Let us denote by φ_1 the angle between the axes Cx_1 and $C\xi_1$ of the "mean" and the König coordinate systems. We then have

$$\gamma_1 = \cos(\varphi_1 - \vartheta), \quad \gamma_2 = -\sin(\varphi_1 - \vartheta), \quad \gamma_3 = 0$$
$$\omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = \varphi_1^{\cdot}$$

When $\varepsilon = 0$, (2.4) yields $\varphi_1^{\cdot} = \text{const}$, and such a motion represents a generating motion for determining the displacement field.

We shall seek the solution of the quasistatic problem of the deformation of a visco-elastic sphere, using the principle of correspondence between the elastic and viscoelastic problem /5, 6/. To do this we replace the shear modulus G in the Laplace transform of the elastic displacement in time, by

$$G_v = E + \eta s + \sum_{l=1}^{M}\frac{G_l s}{s + \tau_l^{-1}}$$

where s is the transformation parameter. Then using the relation /6/

$$G_v^{-1} = \sum_{l=1}^{M+1}\frac{C_l\delta_l}{\delta_l + s} \tag{2.5}$$

(C_l, δ_l are dimensional positive constants) and calculating the original, we obtain the displacement field of the viscoelastic problem.

The solution of the quasistatic problem of the deformation of the elastic planet ($\mathbf{w}(\mathbf{r}, t)$) has the form /3/

$$\begin{gathered}\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 \\ \mathbf{w}_1 = \frac{\rho\omega_3^2}{2G}\mathbf{w}^*(\mathbf{r}), \quad \mathbf{w}_2 = -\frac{3\mu\rho}{2R^3G}O_1^{-1}\mathbf{w}^*(O_1\mathbf{r}) \\ \mathbf{w}_3 = -\frac{\mu\rho(1-2\nu)}{R^3G20(1-\nu)}\left[r^2 - \frac{3-\nu}{1+\nu}r_0^2\right]\mathbf{r} \\ O_1 = \begin{Vmatrix}\sin(\varphi_1-\vartheta) & \cos(\varphi_1-\vartheta) & 0 \\ 0 & 0 & 1 \\ \cos(\varphi_1-\vartheta) & -\sin(\varphi_1-\vartheta) & 0\end{Vmatrix} \\ \mathbf{w}^*(\mathbf{r}) = [(B_1\mathbf{r}, \mathbf{r})B_2 + (B_3\mathbf{r}, \mathbf{r})B_4 + B_5]\mathbf{r}\end{gathered} \tag{2.6}$$

$$\begin{gathered}B_1 = \operatorname{diag}\{b_1, b_1, b_2\}, \quad B_2 = \operatorname{diag}\{1, 1, 0\}, \quad B_3 = \operatorname{diag}\{a_1, a_1, a_2\} \\ B_4 = \operatorname{diag}\{0, 0, 1\}, \quad B_5 = \operatorname{diag}\{c_1, c_1, c_2\} \\ a_1 = 2(3-\nu)\alpha(\nu), \quad a_2 = (1+3\nu)\alpha(\nu) \\ b_1 = -(4 - 3\nu - 5\nu^2)\alpha(\nu), \quad b_2 = -(9 - 8\nu - 5\nu^2)\alpha(\nu) \\ \alpha(\nu) = {}^1/_5(1-\nu)^{-1}(5\nu+7)^{-1} \\ c_1 = \frac{r_0^2(12 - 8\nu - 12\nu^2)}{(1+\nu)(35 - 10\nu - 25\nu^2)}, \quad c_2 = -\frac{r_0^2(3 + 18\nu - 3\nu^2 - 10\nu^3)}{(1+\nu)(35 - 10\nu - 25\nu^2)}\end{gathered}$$

In determining $\varepsilon\mathbf{u}^*(\mathbf{r}, t)$ we shall require the expressions ($f_k^{\pm}(e)$ are power series in e):

$$(1+e\cos\vartheta)^3\cos 2\vartheta = \sum_{k=0}^{\infty} f_k^+(e)\cos k\omega_0 t \tag{2.7}$$

$$(1+e\cos\vartheta)^3\sin 2\vartheta = \sum_{k=0}^{\infty} f_k^-(e)\sin k\omega_0 t$$

$$f_k^{\pm}(e) = (1-e^2)^3(\Phi_k(e) \pm \Phi_{-k}(e))$$

$$\Phi_n(e) = \frac{1}{\pi(1-e^2)^{3/2}}\int_0^{\pi}(1+e\cos v)\cos(n\omega_0 t - 2\vartheta)\,d\vartheta, \quad n \in Z$$

Using the correspondence principle and taking (2.5)-(2.7) into account, we obtain the displacement field of a viscoelastic sphere. After substituting $\varepsilon\mathbf{u}^*(\mathbf{r}, t)$, we can write (2.4) in the standard form

$$\varphi_1^{\cdot} = x, \quad \varphi_2^{\cdot} = \omega_0, \quad x^{\cdot} = \varepsilon F^*(\varphi_1, \varphi_2, x) \tag{2.8}$$

where εF^* is the part of expansion of the right-hand side of Eq.(2.4) linear in ε, solved for $\omega_3^{\cdot} = x^{\cdot}$.

3. Let us average the third equation of (2.8) over the "rapid" variables φ_1 and φ_2. We obtain the equation (here and later, in the final results, we shall make the reverse substitution $\varepsilon\mathbf{u}^* = \mathbf{u}$) describing the evolution of the rotational motion of the planet in the resonant case

$$x^{\cdot} = -k_1\sum_{k=-\infty}^{\infty}\Phi_k^2(e)\sum_{l=1}^{M+1}C_l\delta_l\frac{2x-k\omega_0}{\delta_l^2+(2x-k\omega_0)^2} \tag{3.1}$$

$$k_1 = {}^9/_2\omega_0^4\rho^2C^{-1}\int\{[(b_1-a_1)(x_1^2-x_2^2)+(b_2-a_2)x_3^2+c_2-c_1]\times$$
$$(x_1^2+x_3^2)+2(a_1-a_2+b_1-b_2)x_1^2x_3^2\}\,dx > 0$$

We shall seek the positions of equilibrium x_* of Eq.(3.1) (the stationary rotations of the planet with angular velocity $\omega_3 = x_*$) in the form of a series in powers of the eccentricity

$$x_* = \omega_0\sum_{i=0}^{\infty}\alpha_i e^i \tag{3.2}$$

Let us substitute the expansion (3.2) into the right-hand side of (3.1). Expanding in series and equating the coefficients of e^k $(k = 0, 1, \ldots)$, to zero, we find that the position of equilibrium is unique. Taking into account the first terms in expansion $\Phi_k(e)$ /1, 11/, we obtain the coefficients

$$\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 6\Sigma_1^{-1}\Sigma_2, \quad \alpha_3 = 0 \tag{3.3}$$

$$\alpha_4 = {}^1/_8\Sigma_1^{-1}(25\Sigma_2 + 578\Sigma_3 - 100\alpha_2\Sigma_4)$$

$$\Sigma_1 = \sum_{l=1}^{M+1}\frac{C_l}{\delta_l}, \quad \Sigma_2 = \sum_{l=1}^{M+1}\frac{C_l\delta_l}{\delta_l^2+\omega_0^2}$$

$$\Sigma_3 = \sum_{l=1}^{M+1}\frac{C_l\delta_l}{\delta_l^2+4\omega_0^2}, \quad \Sigma_4 = \sum_{l=1}^{M+1}C_l\delta_l\frac{\delta_l^2-\omega_0^2}{(\delta_l^2+\omega_0^2)^2}$$

An analysis of Eq.(3.1) in the neighbourhood of the position of equilibrium x_* showed that it is asymptotically stable. Thus the angular velocity tends, in the process of evolution, to a stationary value of x_*.

Putting $M = 0$ (the Kelvin-Voigt model) and assuming that

$$\delta_1 = \chi d_1, \; d_1 = \text{const}, \quad \chi \sim \varepsilon^{\delta}, \; {}^1/_2 < \delta < 1 \tag{3.4}$$

we obtain the model used in /3-5/. Under the assumptions made here and taking into account the terms linear in χ, we obtain from (3.2) and (3.3) an expansion in powers of the eccentricity, of the known expression /2/ for the limiting angular velocity obtained under the assumption that the lag angle of the "tidal bulge" is proportional to the angular velocity.

4. In the course of evolution the trajectories of system (2.8) intersect the resonance surfaces (in one-dimensional x space the resonance surface is a point)

$$2x - m\omega_0 = 0, \; m \in Z$$

We introduce, in the neighbourhood of the resonance, new variables given by the formulas /12/

$$x = x_0 + \sqrt{\varepsilon} q, \ \theta = 2\varphi_1 - m\varphi_2, \ x_0 = {}^1/_2 m\omega_0$$

From system (2.8) we obtain, to terms in $\sqrt{\varepsilon}$,

$$\varphi_1^{\cdot} = x_0 + \sqrt{\varepsilon} q, \ \theta^{\cdot} = 2\sqrt{\varepsilon} q, \ q^{\cdot} = \sqrt{\varepsilon} F^* (\varphi_1, \theta, q) \tag{4.1}$$

Let us average the second and third equation of (4.1) over the rapid phase φ_1, and write the resulting system in the form of a single second-order equation

$$\theta^{\cdot\cdot} + k_1 \sum_{k=-\infty}^{\infty} \Phi_k^2(e) \sum_{l=1}^{M+1} C_l \delta_l \frac{(m-k)\omega_0 + \theta^{\cdot}}{\delta_l^2 + ((m-k)\omega_0 + \theta^{\cdot})^2} + k_2 \Phi_m(e) \sin\theta = 0 \tag{4.2}$$

$$k_2 = \frac{3(B-A)}{2C}\omega_0^2$$

describing the evolution of rotational motion of the planet in the neighbourhood of the resonance.

If the inequality

$$k_2 |\Phi_m(e)| > k_1 \left| \sum_{k=-\infty}^{\infty} \Phi_k^2(e) \sum_{l=1}^{M+1} \frac{C_l \delta_l (m-k)\omega_0}{\delta_l^2 + (m-k)^2 \omega_0^2} \right| \tag{4.3}$$

holds, then Eq.(4.2) will have two singularities. The inequality (4.3) represents the condition for the existence of a resonance rotation of the planet with angular velocity $\omega_3 = {}^1/_2 m\omega_0$.

Analysing the singular points of Eq.(4.2) we find that one of them is a saddle, and the other is a focus. When the inequalities (4.4) hold, the focus is asymptotically stable, otherwise it is unstable.

$$\sum_{k=-\infty}^{\infty} \Phi_k^2(e) \sum_{l=1}^{M+1} C_l \delta_l \frac{\delta_l^2 - (m-k)^2 \omega_0^2}{(\delta_l^2 + (m-k)^2 \omega_0^2)^2} > 0 \tag{4.4}$$

In the course of the evolution of the rotational motion of the planet, its angular velocity reaches its resonance value. On approaching resonance we find that three qualitatively different cases are possible depending on the parameters of the orbit and the parameters characterizing the properities of the material:

a) the inequality (4.3) holds with the opposite sign and "capture" into resonance is not possible;

b) condition (4.3) holds and inequality (4.4) holds with the opposite sign; the "probability of capture" /1/ into resonance is equal to zero;

c) conditions (4.3) and (4.4) both hold and some of the trajectories are captured into resonance, since in this case Eq.(4.2) has an asymptotically stable singularity.

In cases b) and c) two trajectories exist, tending in each case to an unstable and a saddle-type singularity (separatrices).

5. Let us compare the results obtained here with the well-known results in /1-4/.

The tidal moment ($\langle T \rangle$) averaged over time, according to Darwin's theory, has the form /1/

$$\langle T \rangle = -k_0 \sum_{k=-\infty}^{\infty} \Phi_k^2(e) \sin\delta(2x - k\omega_0), \quad k_0 = \text{const} \tag{5.1}$$

where $\delta(\omega)$ is the lag angle of the tidal component of the frequency ω. The two, normally used models for $\delta(\omega)$ have the form /1, 2/

$$\sin\delta \sim \delta \sim k'\omega, \ k' = \text{const} \tag{5.2}$$

$$\sin\delta \sim k'' \operatorname{sign}\omega, \ k'' = \text{const} \tag{5.3}$$

In our case the tidal moment averaged over time is given by the right-hand side of Eq. (3.1). Comparing the quantities (5.1) and (3.1) we see that in the present case

$$\sin\delta \sim \sum_{l=1}^{M+1} C_l \delta_l \frac{\omega}{\delta_l^2 + \omega^2} \tag{5.4}$$

We note that relations (5.2) and (5.3) can be regarded as assumptions, while (5.4) was obtained from the exact solution of the quasistatic problem of the deformation of a visco-elastic sphere.

If we put $\delta_l = \chi d_l, \ l = 1, \ldots, M+1$ in (5.4) (as in (3.4)), then taking into account terms linear in χ we obtain a model equivalent to (5.2).

The parameters C_l, δ_l, M can be chosen in such a manner (e.g. using the method of least squares) that the quantity (5.4) will approximate, within the interval of angular velocities used here, the signature model (5.3) with a high degree of accuracy.

The model of the material used here is compared with the models used in /3, 4/ in Sect.3.

Thus the existing models of tidal phenomena can be obtained by choosing the parameters determining the properties of the material in the many-parameter model of viscoelasticity discussed here.

The author thanks A.O. Markeyev for discussing this paper.

REFERENCES

1. GOLDREICH P. and PEEL S., Dynamics of plane rotation. Tides and Resonances in the Solar System. Mir, Moscow, 1975.
2. BELETSKII V.V., Motion of a Satellite Relative to the Centre of Mass in a Gravitational Field. Izd-vo MGU, Moscow, 1975.
3. VIL'KE V.G., Motion of a viscoelastic sphere in a central Newtonian force field. PMM, 44, 3, 1980.
4. VIL'KE V.G., KOPYLOV S.A. and MARKOV YU.G., Evolution of the rotational motion of a viscoelastic sphere in a central Newtonian force field. PMM, 49, 1, 1985.
5. CHRISTENSEN R.M., Theory of Viscoelasticity. Academic Press, N.Y., 1971.
6. BLAND D.R., Theory of LInear Viscoelasticity. Pergamon Press, Oxford, 1960.
7. IL'YUSHIN A.A. and POBEDRYA B.E., Fundamentals of the Mathematical Theory of Thermoviscoelasticity. Nauka, Moscow, 1970.
8. MARKEYEV A.P., On the dynamics of an elastic body in a gravitational field. Kosmich. Issledovaniya. 27, 2, 1989.
9. VASIL'YEVA A.B. and BUTUZOV V.F., Asymptotic Expansions of Solutions of Singularly Perturbed Motions. Nauka, Moscow, 1973.
10. DENISOV G.G. and NOVIKOV V.V., On free motions of an almost spherical deformable solid. Izv. Akad. Nauk SSSR, MTT, 3, 1983.
11. CHERNOUS'KO F.L., Resonance phenomena in the motion of a satellite about its centre of mass. Zhurn. Vychisl. Mat. mat. Fiz., 3, 3, 1963.
12. ZHURAVLEV V.F. and KLIMOV D.M., Applied Methods in the Theory of Oscillations. Nauka, Moscow, 1988.

Translated by L.K.